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AN ABSTRACT FUNCTIONAL DIFFERENTIAL EQUATION AND A RELATED NONLINEAR VOLTERRA EQUATION

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ABSTRACT

We study the existence, uniqueness, regularity and dependence upon data of solutions of the abstract functional differential equation

(1) $$\frac{du}{dt} + Au \ni G(u) \quad (0 \leqq t \leqq T), \quad u(0) = x,$$

where $T > 0$ is arbitrary, A is a given m-accretive operator in a real Banach space X, and $G : C([0,T]; \overline{D(A)}) \to L^1(0,T;X)$ is a given mapping. This study provides simple proofs of generalizations of results by several authors concerning the nonlinear Volterra equation

(2) $$u(t) + b * Au(t) \ni F(t) \quad (0 \leqq t \leqq T),$$

for the case in which X is a real Hilbert space. In (2) the kernel b is real, absolutely continuous on $[0,T]$, $b * g(t) = \int_0^t b(t-s)g(s)\,ds$, and $f \in W^{1,1}(0,T;X)$.

1. Introduction and preliminaries

We study the initial value problem

(1.1) $$\begin{cases} \dfrac{du}{dt} + Au \ni G(u) \quad (0 \leqq t \leqq T), \\ u(0) = x \end{cases}$$

where A is a given m-accretive (possibly multi-valued) operator in a real Banach space X with norm $\|\cdot\|$, and G is a given mapping

(1.2) $$G : C([0,T]; \overline{D(A)}) \to L^1(0,T;X).$$

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(See [2] and [8] concerning the notion of an m-accretive operator and other notation not defined here.) Points of interest will be the existence, uniqueness, regularity, and dependence upon data of solutions of (1.1). The method employed is simple. If $x \in \overline{D(A)}$ and $g \in L^1(0,T;X)$, then the evolution problem

$$\text{(1.3)} \qquad \begin{cases} \dfrac{dv}{dt} + Av \ni g \quad (0 \leqq t \leqq T), \\ \\ v(0) = x \end{cases}$$

has a unique "integral" solution (see [2], [5], or [8]) $v \in C([0,T];\overline{D(A)})$. Let $v = H(g)$ denote this solution. *A solution of* (1.1) *is by definition a function* $u \in C([0,T];\overline{D(A)})$ *such that* $u = H(G(u))$. Under suitable assumptions, some iterate of $K = H(G)$ is a strict contradiction and (1.1) has a unique solution u. Further assumptions allow stronger conclusions, e.g. the solution of (1.1) is Lipschitz continuous or a strong solution (see below). The method adapts easily to generalizations (e.g., the operator $A = A(t)$ depends on time), and to the study of the dependence of u on A, G, and x. The basic idea used here is already found in [5] and exploited for G as in (1.2) in [9].

Much of the motivation for our study of (1.1) lies in the fact that we thereby obtain very simple proofs of generalization of results due to several other authors. MacCamy [15] considers the problem

$$\text{(1.4)} \qquad \begin{cases} \dfrac{du}{dt} + mAu(t) + a * Au(t) = f(t) \quad (0 < t \leqq T), \\ \\ u(0) = x \end{cases}$$

where $m > 0$ is a constant, A is a maximal monotone operator in a real Hilbert space H, and a is a real kernel. We use the notation $a * g(t) = \int_0^t a(t-s)g(s)ds$. Under various additional restrictions, MacCamy transforms the problem (1.4) to one of the form (1.1) by the method of the proof of Proposition 1 below. He then treats the resulting special case of (1.1) via a Galerkin argument (which necessitates further restrictions). Our results concerning (1.1) are directly applicable to problems of the sort discussed in [15]. Moreover, we also obtain generalizations of results of Barbu [1], [3], Londen [13], Gripenberg [11] and Londen and Staffans [14] concerning equations of the form

$$\text{(1.5)} \qquad u(t) + b * Au(t) \ni F(t) \quad (0 \leqq t \leqq T).$$

The Volterra equation (1.5) was treated in a Hilbert space setting by these authors, whereas we obtain results in general Banach spaces by very different and simpler proofs.

Section 2 contains the basic results for (1.1). Applied to the study of (1.5) these results show, among other things, that (1.5) has a unique generalized solution whenever b is absolutely continuous, $b(0)>0$, b' is of bounded variation on $[0,T]$, $f\in W^{1,1}(0,T;X)$ (see below) and $F(0)\in\overline{D(A)}$. This fact is established in Section 3, but we present below the basic connection between (1.1) and (1.5) in the case of strong solutions (which will be defined shortly). Section 4 sketches the relationship between this paper and the existing literature and outlines some generalizations.

First recall that if $\mathcal{J}$ is an interval, then $u\in W^{1,1}(\mathcal{J};X)$ means that there is a function $v:\mathcal{J}\to X$ which is strongly integrable on $\mathcal{J}$ (i.e. $v\in L^1(\mathcal{J};X)$) such that

$$u(t)-u(s)=\int_s^t v(\tau)\,d\tau \quad (t,s\in\mathcal{J})\,;$$

then $u'(t)=v(t)$ a.e. on $\mathcal{J}$. It is also known (e.g. [6, p. 148] or [2, p. 16]) that $u\in W^{1,1}(\mathcal{J};X)$ is equivalent to $u:\mathcal{J}\to X$ being absolutely continuous ($u\in AC(\mathcal{J};X)$) and differentiable a.e. on $\mathcal{J}$. If $u\in AC(\mathcal{J};X)$ and X is reflexive, then u is automatically differentiable a.e. on $\mathcal{J}$.

Definition. A strong solution of (1.1) on $[0,T]$ is a function $u\in W^{1,1}(0,T;X)\cap C([0,T];\overline{D(A)})$ satisfying $u(0)=x$ and $u'(t)+Au(t)\ni G(u)(t)$ a.e. on $[0,T]$.

Definition. Let $b\in L^1(0,T;\mathbf{R})$, $F\in L^1(0,T;X)$. A strong solution u of (1.5) on $[0,T]$ is a function $u\in L^1(0,T;X)$ for which there exists $w\in L^1(0,T;X)$ with $w(t)\in Au(t)$ and $u(t)+b*w(t)=F(t)$ a.e. on $[0,T]$.

Proposition 1. *Let $b\in AC([0,T];\mathbf{R})$, $b'\in BV([0,T];\mathbf{R})$ (i.e. $b':[0,T]\to\mathbf{R}$ is of essentially bounded variation), $F\in W^{1,1}(0,T;X)$ and $b(0)=1$. Let u be a strong solution of (1.5) on $[0,T]$. Then u is a strong solution of (1.1) where*

$$(1.6)\qquad \begin{cases} \text{(i)} & G(u)(t)=f(t)+r*f(t)-r(0)u(t)+r(t)x-u*r'(t),\\ \text{(ii)} & f(t)=F'(t),\\ \text{(iii)} & x=F(0),\\ \text{(iv)} & a=b',\\ \text{(v)} & r\in L^1(0,T;\mathbf{R}) \text{ is defined by } r+a*r=-a,\end{cases}$$

and we have used the notation

$$u * r'(t) = \int_0^t u(t-s)dr(s). \tag{1.7}$$

Conversely, let $r \in BV([0,T];\mathbf{R})$, $f \in L^1(0,T;X)$, $x \in \overline{D(A)}$, *and* G *be given by* (1.6(i)). *If* u *is a strong solution of* (1.1), *then* u *is a strong solution of* (1.5), *where*

$$\begin{cases} \text{(i)} \quad F(t) = x + \int_0^t f(s)ds, \\ \text{(ii)} \quad a + a * r = -r, \\ \text{(iii)} \quad b(t) = 1 + \int_0^t a(s)ds. \end{cases} \tag{1.8}$$

PROOF. Let u be a strong solution of (1.5) on $[0,T]$ and $u + b * w = F$ where $w \in L^1(0,T;X)$, and $w(t) \in Au(t)$ a.e. on $[0,T]$. Since $b \in AC([0,T]:\mathbf{R})$ and $f \in W^{1,1}(0,T;X)$, $u = F - b * w \in W^{1,1}(0,T;X)$ and $u' + b(0)w + b' * w = F'$. Since $b(0) = 1$,

$$w + b' * w = F' - u'. \tag{1.9}$$

Simple facts about Volterra equations imply that if $r, a \in L^1(0,T;\mathbf{R})$ are related via (1.6(iv)), then for $v, w \in L^1(0,T;X)$ we have

$$w + a * w = v \Leftrightarrow v + r * v = w. \tag{1.10}$$

The reader can easily check this (or see [16, Chap. 4]). Hence (1.9) and $r + b' * r = -b'$ imply

$$w = F' - u' + r * (F' - u'). \tag{1.11}$$

Finally, $a = b' \in BV([0,T];\mathbf{R})$ is equivalent to $r \in BV([0,T];\mathbf{R})$ ([4, theor. 7.4]), so

$$r * u'(t) = r(0)u(t) - r(t)u(0) + u * r'(t). \tag{1.12}$$

Since $u(0) = F(0) = x$ and $w(t) \in Au(t)$ a.e., (1.11) and (1.12) show that u is a strong solution of (1.1) with the identifications (1.6).

The converse is proved by reversing the steps. If $u \in W^{1,1}(0,T;X)$ is a strong solution of (1.1) on $[0,T]$, then $G(u)(t) - u'(t) = w(t) \in Au(t)$ a.e. If G is given by (1.6(i)) and (1.12) is used, one finds that

$$(f - u') + r * (f - u') = w,$$

and by using (1.10) that $w + a * w = f - u'$. Integration of this equation yields (1.5) with the identifications (1.8).

REMARK. When considering (1.5) one may reduce to the case $b(0) = 1$, provided that $b(0) > 0$, since A may be replaced by $b(0)A$. Formally, $b * Au = \tilde{b} * \tilde{A}u$, where $\tilde{b} = b(0)^{-1}b$, $\tilde{A} = b(0)A$.

2. Principal results

It is assumed throughout this section that A is m-accretive. The following simple result establishes the existence and uniqueness of solutions of (1.1).

THEOREM 1. *Let $x \in \overline{D(A)}$, $\gamma \in L^1(0, T; \mathbf{R})$ and let $G : C([0, T]; \overline{D(A)}) \to L^1(0, T; X)$ satisfy*

$$
(2.1) \qquad \begin{cases} \|G(u) - G(v)\|_{L^1(0,t;X)} \leqq \int_0^t \gamma(s) \|u - v\|_{L^\infty(0,s;X)}\, ds \\ \text{for } 0 \leqq t \leqq T \text{ and } u, v \in C([0, T]; \overline{D(A)}). \end{cases}
$$

Then (1.1) *has a unique solution $u \in C([0, T]; \overline{D(A)})$.*

We remark that assumption (2.1) implies that the value of $G(u)$ at $t \in [0, T]$ depends only on the restriction of u to $[0, t]$.

Under further assumptions one can obtain greater regularity of solutions of (1.1) than mere continuity. For example:

THEOREM 2. *In addition to the hypotheses of Theorem* 1 *assume that there is a function $k : [0, \infty) \to [0, \infty)$ such that*

$$
(2.2) \qquad \begin{cases} \operatorname{var}(G(u) : [0, t]) \leqq k(R)(1 + \operatorname{var}(u : [0, t])) \\ \text{and } \|G(u)(0+)\| \leqq k(R), \quad 0 \leqq t \leqq T, \end{cases}
$$

whenever $u \in C([0, T]; \overline{D(A)})$ is of bounded variation and $\|u\|_{L^\infty(0,T;X)} \leqq R$. (The variation of a function v over an interval $\mathcal{I}$ is denoted by $\operatorname{var}(v : \mathcal{I})$*.) If $x \in D(A)$, then the solution u of* (1.1) *is Lipschitz continuous on $[0, T]$. If X is also reflexive, then the solution u of* (1.1) *is a strong solution on $[0, T]$.*

Finally, we note that the solution u of (1.1) depends continuously on the "data" A, G, x in the following sense:

THEOREM 3. *Let the assumptions of Theorem* 1 *be satisfied. Let an m-accretive operator A_n in X, a mapping $G_n : C([0, T]; X) \to L^1(0, T; X)$ and $x_n \in \overline{D(A_n)}$ be given for $n = 1, 2, \cdots$. Assume that:*

(2.3) $\begin{cases} \textit{The inequality (2.1) holds with } G \textit{ replaced by } G_n \textit{ for all} \\ u, v \in C([0,T];X), \textit{ with the same } \gamma \textit{ for each } n = 1,2,\cdots; \end{cases}$

$$\lim_{n\to\infty} G_n(u) = G(u) \quad in \quad L^1(0,T;X) \quad for \quad u \in C([0,T];\overline{D(A)}); \tag{2.4}$$

$$\lim_{n\to\infty} (I+\lambda A_n)^{-1} z = (I+\lambda A)^{-1} z \quad for \quad \lambda > 0, z \in X; \tag{2.5}$$

and

$$\lim_{n\to\infty} x_n = x \in \overline{D(A)}. \tag{2.6}$$

Let $u_n \in C([0,T];\overline{D(A_n)})$ *be solutions of* (1.1) *on* $[0,T]$ *with* A *replaced by* A_n, G *replaced by* G_n, x *replaced by* x_n *and let* $u \in C([0,T];\overline{D(A)})$ *be the solution of* (1.1) *on* $[0,T]$. *Then* $\lim_{n\to\infty} u_n = u$ *in* $C([0,T];X)$.

Proof of Theorem 1. Denote the integral solution of $v' + Av \ni g$, $v(0) = x$, $g \in L^1(0,T;X)$ by $v = H(g)$. We seek a fixed point of the map $K : C([0,T];\overline{D(A)}) \to C([0,T];\overline{D(A)})$ defined by $K(u) = H(G(u))$. By properties of integral solutions

$$\|K(u)(t) - K(v)(t)\| \leqq \int_0^t \|G(u)(s) - G(v)(s)\| ds$$

for $0 \leqq t \leqq T$, $u, v \in C([0,T];\overline{D(A)})$. By (2.1) we thus have

$$\|K(u) - K(v)\|_{L^\infty(0,t;X)} \leqq \int_0^t \gamma(s) \|u - v\|_{L^\infty(0,s;X)} ds \quad (0 \leqq t \leqq T). \tag{2.7}$$

Iterating (2.7) one shows by induction that

$$\|K^j(u) - K^j(v)\|_{L^\infty(0,t;X)} \leqq \int_0^t \gamma_j(s) \|u - v\|_{L^\infty(0,s;X)} ds \quad (0 \leqq t \leqq T) \tag{2.8}$$

where

$$\gamma_j(s) = \gamma(s) \int_0^s \gamma_{j-1}(\sigma) d\sigma; \quad j = 2,3,\cdots; \quad \gamma_1 = \gamma. \tag{2.9}$$

Now $\lim_{j\to\infty} \int_0^T \gamma_j(s) ds = 0$, since $\gamma \in L^1(0,T;\mathbf{R})$, and so K^j is a strict contradiction on $C([0,T];\overline{D(A)})$ for sufficiently large j. This establishes the result.

Remark. Theorem 1 is a mild generalization (with the same proof) of [9, lemma 2.1].

PROOF OF THEOREM 2. Define the function $u_0:[0,T]\to X$ by $u_0(t)=x$, $0\leqq t\leqq T$. The proof of Theorem 1 shows that the iterates $u_{n+1}=K(u_n)=H(G(u_n))$, $n=0,1,\cdots$, converge uniformly to the solution u of (1.1) on $[0,T]$. Hence the iterates u_n are uniformly bounded. If $g\in BV([0,T];X)$ in the evolution equation (1.3), then the solution $v=H(g)$ satisfies

$$(2.10)\qquad \begin{cases} \operatorname{var}(v:[0,t])\leqq \|g(0+)-y\|t+\int_0^t \operatorname{var}(g:[0,\tau])\,d\tau \\ \text{for } y\in Av(0)=Ax \quad \text{and for} \quad 0\leqq t\leqq T. \end{cases}$$

In fact (2.10) follows from the stronger inequality

$$(2.11)\qquad \begin{cases} \|v(\xi)-v(\eta)\|\leqq|\xi+\eta|[\|g(0+)-y\|+\operatorname{var}(g:[0,t])] \\ \text{for} \quad y\in Av(0),\quad 0\leqq\xi,\ \eta\leqq t. \end{cases}$$

See, e.g. [2, p. 132] or [5, prop. 1.6]. Thus by (2.2), (2.10) and the uniform boundedness of $\{u_n\}$, there exists a constant c such that

$$(2.12)\qquad \operatorname{var}(u_{n+1}:[0,t])\leqq c\left(1+\int_0^t \operatorname{var}(u_n:[0,\tau])\,d\tau\right)\quad (0\leqq t\leqq T).$$

But then $\operatorname{var}(u_{n+1}:[0,t])\leqq c\exp(ct)$. Thus $\{\operatorname{var}(u_n:[0,T])\}$ and $\operatorname{var}\{G(u_n):[0,T]\}$ are both bounded. By (2.11), $\{u_n\}$ is uniformly Lipschitz continuous. Hence $u=\lim_{n\to\infty}u_n$ is Lipschitz continuous. If X is reflexive, $u\in W^{1,1}([0,T];X)$ follows from the absolute continuity, and u is a strong solution of (1.1) on $[0,T]$.

PROOF OF THEOREM 3. In the proof of Theorem 1 the solution of (1.1) was represented as the unique fixed point of a mapping $K=HG$. H depends on the "data" A and x which we now exhibit explicitly: $H(A,x,g)$ is the integral solution of (1.3) for $x\in\overline{D(A)}$ and $g\in L^1(0,T;X)$. We indicate the dependence of K on A, x, G of (1.1) by

$$K(A,x,G)(u)=H(A,x,G(u)).$$

In the proof of Theorem 1, K^j was a strict contraction for some j. Both j and the contraction constant depend only on the function γ which we assume in (2.3) to be *uniform in n*. Thus, by the argument of Theorem 1, there is a $j>0$ and $0<l<1$ such that if $K_n(u)=K(A_n,x_n,G_n)(u)$, then

$$(2.13)\qquad \begin{cases} \|K_n^j(u)-K_n^j(v)\|_{L^\infty(0,T;X)}\leqq l\|u-v\|_{L^\infty(0,T;X)} \\ \text{for} \quad u,v\in C([0,T];X),\quad n=1,2,\cdots. \end{cases}$$

If $u \in C([0,T];\overline{D(A)})$, $u_n \in C([0,T];\overline{D(A_n)})$ satisfy the equations $K(u) = K(A,x,G)(u) = u$ and $K_n(u_n) = K(A_n,x_n,G_n)(u_n) = u_n$, then using (2.13)

$$\begin{aligned}\|u - u_n\|_\infty &= \|K_n^j(u_n) - K^j(u)\|_\infty \\ &\leqq \|K_n^j(u_n) - K_n^j(u)\|_\infty + \|K_n^j(u) - K^j(u)\|_\infty \\ &\leqq l\|u_n - u\|_\infty + \|K_n^j(u) - K^j(u)\|_\infty,\end{aligned}$$

where $\|\ \|_\infty$ denotes the norm in $L^\infty(0,T;X)$. Thus

$$\|u - u_n\|_\infty \leqq \frac{1}{1-l}\|K_n^j(u) - K^j(u)\|_\infty,$$

and the assertion follows if $\|K_n^j(u) - K^j(u)\|_\infty \to 0$ as $n \to \infty$. For this it suffices to know that $\|K_n(u) - K(u)\|_\infty = \|H(A_n,x_n,G_n)(u) - H(A,x,G)(u)\|_\infty \to 0$ as $n \to \infty$. But the latter follows from [5, prop. 1.23] in view of (2.4), (2.5), (2.6). The proof is complete.

REMARK 2.14. The dependence of the fixed point u of the mapping $K(A,x,G)$ on x and G can be exhibited more precisely. Let $u = K(A,x,G)(u)$, $\hat{u} = K(A,\hat{x},\hat{G})(\hat{u})$ where G, $\hat{G}$ satisfy (2.1). Then by properties of integral solutions and (2.1)

$$\begin{aligned}\|\hat{u}(t) - u(t)\| &\leqq \|x - \hat{x}\| + \int_0^t \|G(u)(s) - \hat{G}(\hat{u})(s)\|\,ds \\ &\leqq \|x - \hat{x}\| + \int_0^t \|G(u)(s) - G(\hat{u})(s)\|\,ds \\ &\quad + \int_0^t \|G(\hat{u})(s) - \hat{G}(\hat{u})(s)\|\,ds \\ &\leqq e(t) + \int_0^t \gamma(s)\|u - \hat{u}\|_{L^\infty(0,s;X)}\,ds,\end{aligned}$$

where

$$e(t) = \|x - \hat{x}\| + \int_0^t \|G(\hat{u})(s) - \hat{G}(\hat{u})(s)\|\,ds. \tag{2.15}$$

Thus by Gronwall's inequality

$$\|\hat{u}(t) - u(t)\| \leqq \|\hat{u} - u\|_{L^\infty(0,t;X)} \leqq e(t) + \int_0^t \exp\left(\int_s^t \gamma(\tau)d\tau\right) e(s)\,ds. \tag{2.16}$$

3. Applications to the abstract Volterra equation

Consider the problem

$$u + b * Au \ni F, \tag{3.1}$$

where B, F are as in Proposition 1 and A is m-accretive on X. Let $\lambda > 0$ and $A_\lambda = \lambda^{-1}(I - (I + \lambda A)^{-1})$ be the Yosida approximation of A. $A_\lambda : X \to X$ is Lipschitz continuous, so a simple contraction argument shows that the approximating problem

$$u_\lambda + b * A_\lambda u_\lambda = F \tag{3.1$_\lambda$}$$

has a unique strong solution u_λ on $[0, T]$ if $b \in L^1(0, T; \mathbf{R})$ and $F \in L^1(0, T; X)$. Our main result for (3.1) is:

THEOREM 4. *Let b, F satisfy the assumptions of Proposition* 1 *and* $F(0) \in \overline{D(A)}$. *Let u_λ be the solution of* $(3.1)_\lambda$ *on* $[0, T]$. *Then* $\lim_{\lambda \downarrow 0} u_\lambda = u$ *in* $C([0, T]; X)$ *where u is the solution of the delay equation* (1.1) *on* $[0, T]$ *of Theorem* 1, *with the identifications* (1.6). *If, moreover,* $F' \in BV([0, T]; X)$ *and* $F(0) \in D(A)$, *then u is Lipschitz continuous on* $[0, T]$. *If X is also reflexive, then* $u \in W^{1,1}(0, T; X)$ *and u is a strong solution of* (3.1).

PROOF. With G given by (1.6) (i) we have

$$\| G(u)(t) - G(v)(t) \| \leqq (| r(0) | + \operatorname{var}(r : [0, t])) \| u - v \|_{L^\infty(0, t; X)} .$$

Hence

$$\| G(u) - G(v) \|_{L^1(0, t; X)} \leqq \int_0^t \gamma(s) \| u - v \|_{L^\infty(0, s; X)} ds, \tag{3.2}$$

with

$$\gamma(s) = | r(0) | + \operatorname{var}(r : [0, s]). \tag{3.3}$$

Since $r \in BV([0, T]; \mathbf{R})$ (because $a = b'$ has this property—see Proposition 1), (3.2), (3.3) imply that G satisfies (2.1). By Proposition 1, u_λ is a strong solution of

$$\frac{du_\lambda}{dt} + A_\lambda u_\lambda \ni G(u_\lambda), \quad u_\lambda(0) = F(0). \tag{3.4}$$

One has $\lim_{\lambda \downarrow 0} (I + \mu A_\lambda)^{-1} z = (I + \mu A)^{-1} z$ for $\mu > 0$, $z \in X$. Thus by Theorem 3 the u_λ converge to u in $C([0, T]; X)$ as desired. If $F' \in BV([0, T]; X)$, (1.6)((i)–(iii)) imply that

$$(3.5)\qquad \begin{cases} \operatorname{var}(G(u):[0,t]) \leqq C(1+\operatorname{var}(u:[0,t])) \\ \\ \|G(u)(0+)\| \leqq C, \quad 0 \leqq t \leqq T, \end{cases}$$

where C is a constant depending on $F(0), F'(0^+), \operatorname{var}(F':[0,T]), r(0+)$, and $\operatorname{var}(r:[0,T])$. Thus Theorem 2 implies that the solution u is Lipschitz continuous on $[0,T]$ if $u(0)=F(0)\in D(A)$. If X is reflexive, u is a strong solution of (1.1) on $[0,T]$, and by Proposition 1, u is a strong solution of (3.1) on $[0,T]$. This completes the proof of Theorem 4.

Observe that if (3.1) has a strong solution u on $[0,T]$ under the assumptions of Theorem 2, then it follows from Theorem 2 and Proposition 1 that $\lim_{\lambda\downarrow 0} u_\lambda = u$ in $C([0,T];X)$. However, whether or not (3.1) has a strong solution, the solutions u_λ of $(3.1)_\lambda$ converge to a limit u as $\lambda \downarrow 0$. We adapt the point of view that this limit is a generalized solution of (3.1):

DEFINITION. *Let the assumptions of Theorem* 4 *be satisfied. Then* $u = \lim_{\lambda\downarrow 0} u_\lambda$ *in* $C([0,T];X)$ *is the generalized solution of* (3.1) *on* $[0,T]$.

The generalized solution is a continuous function of the data b, A, F via Theorem 3. The dependence of the solution on b, F can be estimated explicitly by the method of Remark 2.14. We present below the simpler estimate which results from varying only F.

THEOREM 5. *Let* b *satisfy the assumptions of Theorem* 4 *and* F, $\hat{F}\in W^{1,1}(0,T;X)$, $F(0)$, $\hat{F}(0)\in\overline{D(A)}$. *Let* $u,\hat{u}\in C([0,T];\overline{D(A)})$ *be the generalized solutions on* $[0,T]$ *of the equations* $u+b*Au\ni F$ *and* $\hat{u}+b*A\hat{u}\ni\hat{F}$ *respectively. Then*

$$\|u(t)-\hat{u}(t)\| \leqq e(t) + \int_0^t \exp\left(\int_s^t \gamma(\tau)d\tau\right) e(s)ds \quad (0\leqq t\leqq T),$$

where

$$\begin{cases} e(t) = c_1\|F(0)-\hat{F}(0)\| + c_2\displaystyle\int_0^t \|F'(s)-\hat{F}'(s)\|ds, \\ c_1 = 1+\|r\|_{L^1(0,T;\mathbf{R})}, \qquad c_2 = 1+\|r\|_{L^\infty(0,T;\mathbf{R})}, \\ \gamma(s) = |r(0+)| + \operatorname{var}(r:[0,s]), \end{cases}$$

and r *is determined by* $r+b'*r=-b'$.

Theorem 5 is proved by substituting into (2.15), (2.16) and estimating. Here G, $\hat{G}$ are given by (1.6)(i) with $f=F'$, $x=F(0)$ and $f=\hat{F}'$, $x=\hat{F}(0)$ respectively; γ

is determined from (3.2), (3.3). Theorem 5 implies that the mapping: $W^{1,1}(0,T;X) \ni F \to$ the generalized solution u of (3.1) on $[0,T]$ is Lipschitz continuous on the set of $F \in W^{1,1}(0,T;X)$ satisfying $F(0) \in \overline{D(A)}$. Thus if X is reflexive, the generalized solution of (3.1) may be regarded as the unique limit of strong solutions.

4. Connections with other research and generalizations

Concerning the existence and uniqueness of solutions of the Volterra equation (3.1), Theorem 4 generalizes results of Barbu [1], Londen [13] and MacCamy [15], all for the case $X = H$ a real Hilbert space, and $Au = \partial\varphi(u)$, where $\partial\varphi$ denotes the subdifferential of a function $\varphi : H \to (-\infty, \infty]$ which is convex, l.s.c., and proper, and of Gripenberg [11, theor. 1] for the case in which $X = H$ and A is a maximal monotone operator on H. Barbu [1] and Gripenberg [11, theor. 2] consider certain cases when the kernel b is operator-valued. Our assumptions concerning the kernel b and the function F are closest to those of [11] and [13]. In Barbu [1] the kernel b is assumed to be of positive type which, while less general in some respects than the kernels of [11], [13] and this paper does permit the possibility that $b(0+) = +\infty$. Londen and Staffans [14] also study (3.1) in the case $X = H$, $Au = \partial\varphi(u)$, and they relax the assumption of [11] and [13] that $b' \in BV[0,T]$; they require instead that b' satisfies a frequency domain condition which holds automatically if $b' \in BV[0,T]$. Theorem 5 has not been considered by any of these authors.

We should mention how the case in which $A = \partial\varphi$ is treated in our context. From [6, theor. 3.6] it follows at once that in this case the solution u of (3.1) provided by Theorem 4 satisfies $u' \in L^2(0,T;H)$ if only $F(0) \in D(\varphi)$ and $F' \in L^2(0,T;H)$. If $F(0) \in \overline{D(\varphi)}$ and $F' \in L^2(0,T;H)$, then we have $tu' \in L^2(0,T;H)$. Moreover, $b' \in BV([0,T];H)$ can be relaxed to (for example) the assumption that $b \in AC([0,T];H)$ and $b' \in BV([0,T_0];H)$ for some $T_0 > 0$ by use of a continuation argument as in [13].

Theorem 4 can be used to strengthen a result of Clément and Nohel [7, theor. 5] concerning the positivity of solutions of the Volterra equation (3.1).

If $A(t)$ is a m-accretive operator for almost all $t \in [0,T]$, $\overline{D(A(t))} = D$ is constant a.e. and $g \in L^1(0,T;X)$, it is known [10] that the evolution problem

$$(4.1) \qquad \begin{cases} \dfrac{dv}{dt} + A(t)v \ni g(t) \\ \\ u(0) = x \end{cases} \qquad (0 \leqq t \leqq T),$$

has a "solution" $v \in C([0,T];D)$ if $x \in D$, provided only that either

$$(4.2)\quad \begin{cases} \text{There exists a Banach space } Y,\ h \in L^1(0,T;Y), \text{ and a continuous function } L:[0,\infty)\to[0,\infty) \text{ such that} \\ \|A_\lambda(t)x - A_\lambda(s)x\|_X \leqq \|h(t)-h(s)\|_Y L(\|x\|_X) \\ \text{for all } \lambda > 0, x \in X \text{ and almost all } s,t \in [0,T], \end{cases}$$

or

$$(4.3)\quad \begin{cases} \text{There exist } h, L \text{ as in (4.2), } h \in BV([0,T];Y) \text{ such that} \\ \|A_\lambda(t)x - A_\lambda(s)x\|_X \leqq \|h(t)-h(s)\|_Y L(\|x\|_X)(1+\|A_\lambda(s)x\|) \\ \text{for } \lambda > 0,\ x \in X \text{ and almost all } s,t \in [0,T], \end{cases}$$

where $A_\lambda(t) = \lambda^{-1}(I-(I+\lambda A(t))^{-1})$. Moreover, if $v = H(g)$ is the solution of (4.1), then

$$(4.4)\quad \|H(g)(t) - H(\hat{g})(t)\| \leqq \int_0^t \|g(s)-\hat{g}(s)\|\,ds \quad (0 \leqq t \leqq T),$$

where $g, \hat{g} \in L^1(0,T;X)$. Since the property (4.4) of H was all we needed to find fixed points of the map $K = HG$ in Section 2, we can therefore solve the more general delay equation

$$\frac{du}{dt} + A(t)u \ni G(u), \qquad u(0) = x \quad (0 \leqq t \leqq T)$$

where $G(u)$ satisfies (2.1), (2.2), and also the Volterra equation

$$u + b * A(t)u \ni F \quad (0 \leqq t \leqq T),$$

as in Sections 2 and 3. The notion of "solution" of (4.1) is more complex than if A is independent of t, and the technical details concerning (4.1) are otherwise too complicated to warrant more precision here. Thus we remark only that (4.2) or (4.3) suffice for a good existence theory and, given knowledge of (4.1), the proofs are the same. See [10] and [17] concerning (4.1). In particular, one easily generalizes the results of Gripenberg [11, theor. 2].

Another type of generalization arises if we relax the assumptions concerning G in Theorem 1. For example let G be only "locally Lipschitz" in the following sense:

$$
(4.5)\qquad \begin{cases} \text{Let } x \in \overline{D(A)}, u_0(t) = x, \quad 0 \leqq t \leqq T, \quad R > 0, \\ \mathcal{M} = \{u \in C([0,T]; \overline{D(A)}) : \|u - u_0\|_{L^\infty(0,T;X)} < R\}, \\ \qquad\qquad\qquad\qquad \text{and} \quad \gamma \in L^1(0,T;\mathbf{R}^+). \\ \text{Assume that } G : \mathcal{M} \to L^1(0,T;X), \text{ satisfies} \\ \|G(u) - G(v)\|_{L^1(0,t;X)} \\ \qquad \leqq \int_0^t \gamma(s) \|u - v\|_{L^\infty(0,s;X)} ds \quad \text{for} \quad u, v \in \mathcal{M}. \end{cases}
$$

Then we can prove the following "local theorem" for the delay equation (1.1).

THEOREM 6. *Let* (4.5) *be satisfied. Then there is a* T_0, $0 < T_0 \leqq T$ *and a* (*unique*) $u \in \mathcal{M}$ *such that*

$$
(4.6)\qquad \frac{du}{dt} + Au \ni G(u), \qquad u(0) = x, \quad 0 \leqq t \leqq T_0.
$$

PROOF. Let $w_0 \in C([0,T]; \overline{D(A)})$ be the solution of $w' + Aw \ni G(u_0)$, $w(0) = x$. Choose $T_0 > 0$ such that

$$
(4.7)\qquad \begin{cases} \|w_0(t) - x\| \leqq R/2 \quad (0 \leqq t \leqq T_0), \\ \int_0^{T_0} \gamma(s) ds \leqq 1/2. \end{cases}
$$

Let $v \in \mathcal{M}$ and u be the solution of

$$
\frac{du}{dt} + Au \ni G(v), \quad u(0) = x.
$$

Then using (4.5), (4.7) one has

$$
\begin{aligned} \|u(t) - x\| &\leqq \|u(t) - w_0(t)\| + \|w_0(t) - x\| \\ &\leqq \int_0^t \|G(v)(s) - G(u_0)(s)\| ds + \tfrac{1}{2}R \\ &\leqq R \int_0^t \gamma(s) ds + \tfrac{1}{2}R \leqq R, \quad (0 \leqq t \leqq T_0). \end{aligned}
$$

The solution operator $v \to HG(v) = u$ thus maps the set $\mathcal{M}$ into

$$
\{u \in C([0,T]; \overline{D(A)}) : \|u(t) - u_0(t)\| \leqq R \quad \text{for} \quad 0 \leqq t \leqq T_0\}.
$$

Modify this mapping by setting

$$K_{T_0}(v) = \begin{cases} HG(v)(t) & \text{if} \quad 0 \leqq t \leqq T_0 \\ HG(v)(T_0) & \text{if} \quad T_0 < t \leqq T. \end{cases}$$

$\mathcal{M}$ is now invariant under K_{T_0} and

$$\| K_{T_0}(u) - K_{T_0}(v) \|_{L^\infty(0,t;X)} \leqq \int_0^t \gamma(s) \| u - v \|_{L^\infty(0,s;X)} ds \quad (u, v \in \mathcal{M}).$$

Therefore, K_{T_0} has a unique fixed point u which is a solution of (4.6) on $[0, T_0]$. This completes the proof of Theorem 6.

Finally, if assumption (2.1) is relaxed to

$$\| G(u) - G(v) \|_{L^1(0,T;X)} \leqq M \| u - v \|_{L^\infty(0,T;X)}, \tag{4.8}$$

so that $G(u)(t)$ can also depend on the values $u(\tau)$ for $\tau > t$ (which (2.1) does not allow), then the technique of Theorem 1 shows that equation (1.1) with G replaced by εG has a unique solution whenever $|\varepsilon| M < 1$.

We also wish to point out that the method developed in Sections 2 and 3 can be used to study the nonconvolution Volterra equation

$$u(t) + \int_0^t b(t,s) A u(s) ds \ni F(t) \quad (0 \leqq t \leqq T), \tag{4.9}$$

where A is m-accretive on X and $F \in W^{1,1}([0,T]; X)$. Concerning b we assume that it is defined on the region $T = \{(t,s) : 0 \leqq s \leqq t \leqq T\}$, that $b(t,s)$ is as smooth as required for the calculations which follow, and that $b(t,t) > 0$ $(0 \leqq t \leqq T)$.

Differentiating (4.9) one obtains

$$\frac{du}{dt}(t) + b(t,t) A u(t) + \int_0^t \frac{\partial b}{\partial t}(t,s) A u(s) ds \ni F'(t); \; u(0) = F(0). \tag{4.10}$$

Putting $\varphi(t) = b(t,t) > 0$, $\tau = \Phi(t) = \int_0^t \varphi(s) ds$ $(0 \leqq t \leqq T)$, $\psi(\tau) = \varphi(\Phi^{-1}(\tau))^{-1} F'(\Phi^{-1}(\tau))$ $(0 \leqq \tau \leqq \Phi(T))$, and defining $v(\tau) = u(\Phi^{-1}(\tau))$, $c(\tau, \sigma) = b(\Phi^{-1}(\tau), \Phi^{-1}(\sigma))$, an elementary calculation shows that (4.10) is transformed to the equivalent initial value problem

$$\begin{aligned} & \frac{dv}{d\tau} + Av(\tau) + \int_0^\tau \frac{\frac{\partial c}{\partial \tau}(\tau, \sigma)}{c(\sigma, \sigma)} Av(\sigma) d \ni \psi(\tau) \quad (0 \leqq \tau \leqq \Phi(T), \\ & v(0) = x = F(0). \end{aligned} \tag{4.11}$$

Let

$$k(\tau,\sigma)=\frac{\frac{\partial c}{\partial \tau}(\tau,\sigma)}{c(\sigma,\sigma)},$$

and define the associated resolvent kernel $r(\tau,\sigma)$ by the equation ([17, chap. IV]):

$$r(\tau,\sigma)+\int_\sigma^\tau k(\tau,\xi)r(\xi,\sigma)d\xi = -k(\tau,\sigma) \quad (0 \leqq \sigma \leqq \tau \leqq \Phi(T)). \tag{4.12}$$

A calculation analogous to that of Proposition 1 shows that the problem (4.11) is (under suitable assumptions) equivalent to the initial value problem

$$\frac{dv}{d\tau}+Av \ni G(v) \quad (0 \leqq \tau \leqq \Phi(T); v(0)=x), \tag{4.13}$$

where

$$\begin{aligned} G(v)(\tau) &= \psi(\tau)+\int_0^\tau r(\tau,\sigma)\psi(\sigma)d\sigma - r(\tau,\tau)v(\tau) \\ &\quad + r(\tau,0)x + \int_0^\tau v(\tau-\sigma)d_\sigma r(\tau,\sigma) \quad (0 \leqq \tau \leqq \Phi(T)). \end{aligned} \tag{4.14}$$

The Stieltje's integral in (4.14) is well defined if $r(\tau,\sigma)$ is of bounded variation with respect to σ on $0 \leqq \sigma \leqq \tau$, uniformly in τ on $0 \leqq \tau \leqq \Phi(T)$. Under this assumption it is clear that G defined in (4.14) satisfies the estimate (3.2) with $\gamma(s)=|r(s,s)|+\sup_{0\leqq\tau\leqq\Phi(T)} \operatorname{var}(r(\tau,\sigma):0 \leqq \sigma \leqq \tau)$. Thus one may apply the theory developed in Section 2 and arrive at analogues of Theorems 4 and 5 for the nonconvolution equation (4.9). We shall not pursue this topic further.

Finally, we remark that if $X=\mathbf{R}$ and A is a nondecreasing (continuous) function from $\mathbf{R}$ to $\mathbf{R}$ in (3.1), and if $F \in AC([0,T])$, J. J. Levin [12, theor. 1′] has obtained by a different method a result similar to the one to which our Theorem 4 reduces for this case.

Added in Proof. Since the completion of this paper we have learned that G. Gripenberg (*On a linear Volterra integral equation in Banach space*, Helsinki Univ. of Tech. Report—HTKK-MAT-A95, 1976) has extended the results of [11]. He obtains results related to our Theorem 4 by more complicated arguments. We also note that further information concerning the nonconvolution equation (4.9) is now available in G. Gripenberg, Helsinki Univ. of Tech. Report—HTKK-MAT-A105 (1977), and C. Rennolet, Ph.D. thesis, Univ. of Wisconsin, Madison, August 1977.

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